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Foliations on 3-manifolds which are the classifying space of themselves

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Abstract. We classify the closed foliated 3-manifolds M, with codimension one foliations of nonexponential growth and which are homotopy equivalent to their classifying space $B\Gamma$. Then we construct arbitrarily "large" manifolds with foliations with of any growth type and satisfying $\pi_1(M) = \pi_1(B\Gamma)$.

Keywords: Foliations, classifying space.

Mathematical subject classification: 57R30, 57R32.

1 Introduction

Given a manifold M with a foliation $\mathcal{F}$ we associate to the holonomy pseudogroup Γ of $\mathcal{F}$ the classifying space $B\Gamma$ of Haefliger (see [Hae]). This space is in general of infinite dimension and can be seen as a foliated space by a foliation Υ whose holonomy covering of each leaf is contractible. Moreover, there exists a map $f : M \to B\Gamma$ such that $f^*\Upsilon = \mathcal{F}$ which is unique up to homotopy equivalence. The interest in understanding this space comes from the fact that the homotopy, homology or cohomology groups of the classifying space $B\Gamma$ are invariants of the transverse structure. The characteristic classes of foliations $\mathcal{F}$ (e.g. the Godbillon-Vey invariant) come from universal classes defined in the cohomology of $B\Gamma$.

If $\mathcal{F}$ is such that the holonomy covering of each leaf is contractible, then up to homotopy equivalence, M is its classifying space. This is the case, for example, when M is a 3-manifold and $\mathcal{F}$ is a foliation of M by surfaces other than the sphere or the projective plane, and such that the holonomy of each leaf is an injective representation of the fundamental group of this leaf. That will be one

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of the subjects of this paper. For all the assertions previously made, we refer to [Hae].

The most accessible invariant associated to $B\Gamma$ is its fundamental group $\pi_1(B\Gamma)$, because it has an explicit definition: it is the quotient of $\pi_1(M)$ by the normal sub-group $\mathcal{L}$, generated by the free homotopy classes of loops contained in leaves, and of trivial holonomy [Sal]. In [Gus] this group was characterized when M^n is a closed manifold of dimension n and $\mathcal{F}$ is a C^r $(r \geq 2)$ transversely orientable codimension one foliation almost without holonomy. In this case, it is isomorphic to the fundamental group of certain graph of groups (G,Y) where the groups of edges and vertices are abelian. The graph Y is obtained starting from the decomposition in models for codimension one foliations (see sec.2).

Another purpose of this work is to understand foliations which satisfy $\pi_1(M) = \pi_1(B\Gamma)$. Let us note that in this case, any loop freely homotopic to a loop of trivial holonomy is trivial in M. In particular, any leaf L without holonomy is completely compressible, i.e., $\pi_1(L)$ is mapped to zero in $\pi_1(M)$. So this situation when M has dimension 3 and $\mathcal{F}$ has codimension one is the opposite of that of foliations without Reeb components, since in the latter case the fundamental groups of leaves are injectively mapped into the fundamental group of the manifold.

Let M be a 3-manifold with a transversely orientable codimension one foliation. We say that $(M, \mathcal{F})$ satisfies properties P_1, respectively P_2 if:

[P_1] M is its own classifying space, i.e., the covering of holonomy of each leaf is contractible.

[P_2] $\pi_1(M) = \pi_1(B\Gamma)$.

Note that the second property says that any loop freely homotopic to a loop of trivial holonomy in a leaf bounds a disc in M, and thus, P_1 implies P_2. Clearly the converse is not true. On the other hand, if $\mathcal{F}$ is without holonomy, the two properties are equivalent (indeed, since the trivial foliation of $S^2 \times S^1$ is ruled out, the leaves are planes and M is the torus T^3 [Ro]). Thus we conclude that, in this case, these properties impose strong restrictions on the topology of the leaves and also on the manifold, so it is natural to expect the same thing for foliations with more complicated transverse structure. Nevertheless, we will see that this is not the case, at least if property P_2 holds.

We shall consider these properties in the following cases:

1) *Foliations with nonexponential growth satisfying property* P_1

2) *Foliations satisfying property* P_2.

Concerning the first problem, we prove the following:

Theorem 1. *Let M^3 be an orientable closed manifold, and $\mathcal{F}$ a codimension one foliation, transversely orientable with nonexponential growth. If $\mathcal{F}$ satisfies P_1, then the depth k of $\mathcal{F}$ is finite, and it is equal to 0 or 1. If k=0, then $M = T^3$ foliated by planes. If k=1 (thus almost without holonomy) and $\mathcal{F}$ does not have Reeb components, then M^3 is a fibration over S^1 with fiber T^2, and the foliation is (up to conjugation) as in Example 1 (see section 3). If $\mathcal{F}$ has a Reeb component, then $(M, \mathcal{F})$ is (up to smash a product $T^2 \times [0, 1]$ foliated as in Example 1) obtained by gluing two copies of $D^2 \times S^1$ by a diffeomorphism of the boundary, each one with a Reeb foliation.*

Then, starting from a certain number of examples (Section 3), we show that property P_2 does not impose any restriction on the growth of the leaves. More precisely, we prove:

Theorem 2. *Given $n, g \in \mathbb{N}$, there exists a manifold M^3 with a transversely orientable codimension one foliation $\mathcal{F}$ of class C^r ($r \geq 2$) satisfying $\pi_1(M) = \pi_1(B\Gamma)$ with any number of compact surfaces Σ_g of genus g, and such that the fundamental group of the associated graph Y is the free group $\mathbb{F}_n$, which is injected into $\pi_1(M)$. Moreover, for each $k \in \mathbb{N}$, there exist saturated open sets $U_1, \ldots, U_k$ such that $\mathcal{F}|_{U_i}$ has one of the following growth types:*

a) polynomial growth;

b) quasipolynomial but nonpolynomial growth;

c) nonexponential but not quasipolynomial growth, or

d) exponential growth.

This paper is structured as follows. In Section 2, we recall what models of codimension one foliations are. We devote Section 3 to present several examples of foliated manifolds satisfying property P_2. We prove Theorem 1 in Section 4 and Theorem 2 is proved in Section 5.

2 The models of codimension one foliations and the graph Y

In this section $\mathcal{F}$ is a C^2 transversely orientable codimension one foliation on a closed manifold M of dimension n. If the set of compact leaves $C(\mathcal{F})$ is nonempty, then $C(\mathcal{F})$ is a saturated compact subset of M.

Following [F.S], two compact leaves F and F' are *equivalent* if there exists an immersion $h : F \times [a, b] \to M$ satisfying the following conditions:

1. For every $t \in [a, b]$ the restriction of the map h to $F \times \{t\}$ is an embedding of F in M,
2. $h(F \times \{a\}) = F$ and $h(F \times \{b\}) = F'$,
3. For each $x \in F$, the path $h_x : (a, b) \to M$ is transverse to $\mathcal{F}$.

We observe that the condition 3 implies that the foliation $h^*(\mathcal{F})$ is defined by a suspension. The following results are proved in [F.S]:

a) $\mathcal{F}$ has finitely many equivalence classes of compact leaves;

b) Let Λ be the set of all immersions which realize some equivalence of compact leaves equivalent to F. Then

$$\bigcup_{h \in \Lambda} h(F \times [a, b])$$

is a compact subset of M, saturated by $\mathcal{F}$ and contains all the compact leaves equivalent to F. This compact set will be referred as the support of F and it will be denoted by $\mathrm{supp}[F]$;

c) There exist $h_0 \in \Lambda$ such that $\mathrm{supp}[F] = h_0(F \times [a, b])$;

d) If F and L are not equivalent compact leaves of $\mathcal{F}$, then $\mathrm{supp}[F] \cap \mathrm{supp}[L] = \emptyset$.

If we denote

$$\mathcal{A} = \bigcup_{F \in C(\mathcal{F})} \mathrm{supp[F]},$$

then by a) and c) above $\mathcal{A}$ has finitely many number of connected components. We see that the complement $M - \mathcal{A}$ is a finite union of connected saturated open sets $U_1, \ldots, U_k$ each one without compact leaves. Moreover, for each $i \in \{1, 2, \ldots, k\}$ we can find a compact manifold with boundary, V_i, and a differentiable immersion h_i of V_i in M with the following properties:

i) h_i is a diffeomorphism of the interior of V_i to U_i;

ii) h_i is a diffeomorphism of each connected component of the boundary of V_i to a compact leaf of $\mathcal{F}$ contained in the closure $\overline{U_i}$ of U_i;

Definition 1. *We call a model of type 1 each one of the connected components of $\mathcal{A}$; and each connected components of $M - \mathcal{A}$ is called model of type 2.*

Now, we define a *graph* Y associated to $(M, \mathcal{F})$ as follows: its vertices are the models of type 2; its edges are the models of type 1. Observe that if $F \in C(\mathcal{F})$ is such that $\mathrm{supp}[F] = F$, then F is a common boundary of two models of type 2, all attached to the vertices in an obvious way or, alternatively, two components of the boundary of one of these models can be identified in M, which gives a cycle in Y with only one vertex and one edge. This situation is depicted in Figure 1 below.

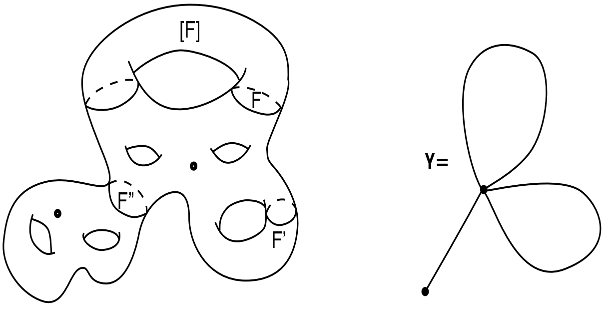


Figure 1:

3 Examples

In this section, we give some examples of closed 3-manifolds with foliations which satisfy property P_2. In what follows, "*make Reeb surgery along a closed transversal curve* γ" means to remove a tubular neighbourhood of γ and then spiral the leaves along to the boundary.

Example 1. Consider $T^2 \times [0, 1]$ with a foliation transverse to the $[0, 1]$ factor and tangent to the boundary, such that the holonomy of compact leaves (tori) has rank 2. So the open leaves are planes. Now, let us identify the two components of the boundary by a diffeomorphism. We thus obtain a fibration over S^1 with fiber T^2 with foliations satisfying P_2.

Example 2. Given T^3 with a foliation by planes, choose a finite number of disjoint closed transversals and do Reeb surgery. Then we obtain a compact manifold M with boundary, where each open leaf is homeomorphic to a plane with countable disjointed open discs removed, and gluing a half-cylinder $S^1 \times [0, \infty)$ along each boundary component. These cylindrical ends are proper and accumulate each one over a component of the boundary, while the end of the plane is dense in $\overset{\circ}{M}$. Now, gluing a Reeb component along each boundary component of M, in order to eliminate the generator of the fundamental group of the proper ends which adhere to the respective boundary component of M, one obtains a closed manifold with an almost without holonomy foliation which satisfies P_2. The associated graph Y is as in Figure 2.

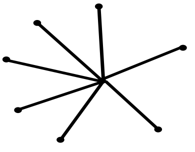


Figure 2:

Example 3. This example is a modification of an example in [C.C2] in a different context. Consider on $D^2 \times S^1$ the foliation obtained in the following way: On $T^2 \times [0, 1]$ let us take the dense foliation by planes whose holonomy of each boundary component has rank two. Then, glue a Reeb component to the boundary $T^2 \times \{1\}$.

We denote by α and β the generators (of the fundamental group) of $T^2 \times \{0\}$, null homotopic in $D^2 \times S^1$ and non null homotopic respectively. Now, we make the Reeb surgery along a closed transversal curve γ freely homotopic to α. So we obtain a compact manifold V with two boundary components, $T_1 = T^2 \times \{0\}$ and T_2, obtained by Reeb surgery (see Figure 3).

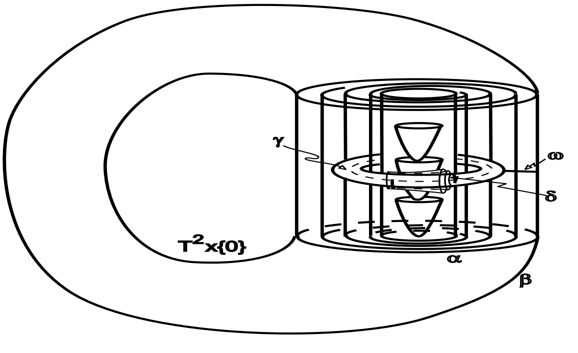


Figure 3:

We denote by δ and the same γ the generators of the fundamental group of the boundary component T_2 (see Fig. 3). Now let us glue the two components of the boundary of V by identifying α and β of T_1 respectively with δ and γ of T_2 to obtain a closed manifold M. If the direction of Reeb surgery is well chosen, the loop ω (see Fig. 3) is transverse to the foliation and again the open leaves are as in Example 2. Also let us note that the loops of trivial holonomy are all freely homotopic to the loop δ (or to some power of δ) of the compact leaf T_2, which coincides with T_1 after identification. Then they are all trivial in $\pi_1(M)$ and so $(M, \mathcal{F})$ satisfies P_2. Observe that the manifold M is homeomorphic to $S^1 \times S^2$, where the factor S^1 is represented by the transverse curve ω (see [C.C2]). The graph Y is depicted in Figure 4.

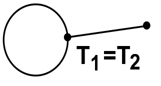


Figure 4:

Also note that, instead of taking dense foliations by planes of $T^2 \times [0, 1]$ we could have taken foliations by cylinders, where the ends of the cylinder spiral in opposed directions around each boundary component. By doing so, we would obtained a manifold M with a proper foliation.

In the examples above, the associated graphs have at most one cycle. Now we will give an example where the graph Y has two cycles and one Reeb component.

Example 4. Let us fix initially the manifold with boundary $V_1 = V$ of Example 3 with boundary components T_1 and T_2 and of another side, consider a copy of $T^2 \times [0, 1]$ with dense foliations by planes tangent to the boundary. Let us make the Reeb surgery along a closed transverse curve λ homotopic to the parallel β of $T_3 = T^2 \times \{0\}$. The manifold obtained V_2 has three boundary components, namely, T_3, the torus T_4, boundary of the tubular neighbourhood, and $T_5 = T^2 \times \{1\}$ (see Figure 5). If we denote by θ and λ respectively the meridian and the parallel of T_4, let V be the manifold obtained gluing V_1 and V_2 by their boundaries in the following way: we glue $T_1 \in \partial V_1$ with $T_3 \in \partial V_2$ by identifying the meridian α of T_1 with the parallel β of T_3 and the parallel β of T_1 with the meridian α of T_3; in the same way, we glue $T_2 \in \partial V_1$ with $T_4 \in \partial V_2$ identifying the meridian δ of T_2 with the parallel λ of T_4 and the parallel λ of T_2 with the meridian θ of T_4.

The manifold V, with the foliation $\mathcal{F}'$ obtained, has only one boundary component T_5. It satisfies P_2 because in V_1 the loops of trivial holonomy (except those in the Reeb component, which are trivial in $\pi_1(V)$) are all homotopic to δ or to some power of δ. Furthermore δ has been identified with λ, which is homotopic to β (the parallel of T_3). Let us note that in V, β is identified with

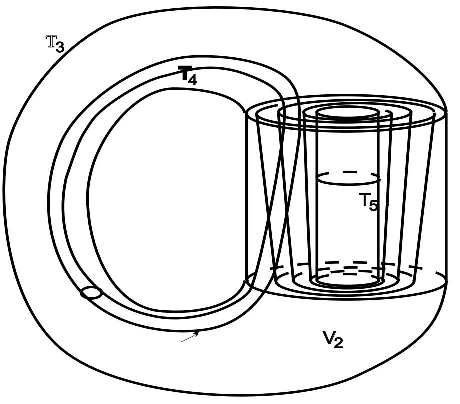
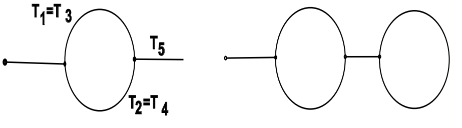


Figure 5:

α (meridian of T_1), therefore it is trivial in $\pi_1(V)$. In the same way, in V_2 the loops of trivial holonomy are all homotopic to θ or to some power of θ, and θ in V is identified with γ (parallel of T_2), which is trivial in $\pi_1(V)$. The associated graph with $(V, \mathcal{F}')$ is as in Figure 6a.

Now, if we denote by θ_1 and θ_2 respectively the meridian and the parallel of T_5, let N be the manifold obtained from V_2 above, by gluing T_3 with T_5 an identifying their meridians and parallels. Now let us glue V and N by their boundaries identifying the meridian θ_1 of $T_5 = \partial V$ with the parallel λ of $T_4 = \partial N$ and the parallel θ_2 of $T_5 = \partial V$ with the meridian θ of $T_4 = \partial N$. But in V, θ_2 is homotopic to $\beta = \alpha \simeq 0$. Again, the foliated manifold $(M, \mathcal{F})$ obtained satisfies P_2 and is such that all noncompact leaves are planar surfaces, each one of them with countable proper ends, and only one end is locally dense. The associated graph is as in Figure 6b.



(a) (b)

Figure 6:

In the examples above, the noncompact leaves are planar surfaces and the compact leaves are tori. In the next example the compacts leaves are tori and compacts surfaces of genus $g \geq 2$. The noncompact leaves which adhere to those surfaces have infinite genus.

Example 5. In Example 3, instead of taking the foliation of $T^2 \times [0, 1]$ by dense planes, let us take the foliation by proper cylinders which accumulate on each boundary component. Let us make then the same operations to obtain the manifold M of the Example 3. Let us note that we can take the transverse curve γ so that it meets each leaf in a single point. Thus, in $M \approx S^1 \times S^2$, the noncompact leaves which are not contained in the Reeb component have three ends $\epsilon_1, \epsilon_2, \epsilon_3$, and the foliation contains only two compact leaves: $T_1 = T_2$ and the boundary of the Reeb component.

Now let us remove the interior of a small tubular neighbourhood of the transverse curve ω and make the double along the boundary. By this operation the toric leaf T_1 becomes a surface of genus 2. Let us note that, before making the double, each open leaf (out of the Reeb component) has two ends, say ϵ_1 (which adhere to T_1 along γ), ϵ_2 (which adhere to T_1 along α) which intercepts ω in an infinite discrete set $\{p_{1n}\}_{n\in\mathbb{N}}$, respectively $\{p_{2n}\}_{n\in\mathbb{N}}$. Thus, by making the double we are gluing the leaves L, L' (in each copy used in the double) along the circles C_{in} $C_{in}{}'$ which are the boundary of $L - \{p_{in}\}$, respectively $L' - \{p_{in}\}$. In particular, the leaves have infinite genus (see Figure 8). The manifold obtained is still homeomorphic to $S^1 \times S^2$ because we remove the interior of a solid torus and glue another solid torus. Let us note that the compact leaf Σ_2 is completely compressible in M, i.e., if $i : \Sigma_2 \to M$ is inclusion, then $i_*(\pi_1(\Sigma_2)) = 0$.

Let us indicate α, β, α', β' the generators of $\pi_1(\Sigma_2)$ and θ as in Figure 7.

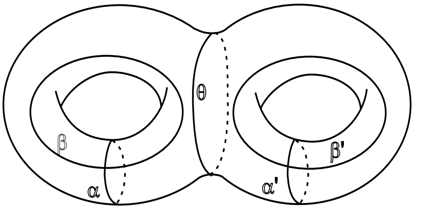


Figure 7:

If $(\epsilon_{1n}, {\epsilon'}_{1n})$, $(\epsilon_{2n}, {\epsilon'}_{2n})$ are the loops indicated in Figure 8, then they are freely homotopic to (α, α'), respectively (β, β'), hence they are trivial in $\pi_1(M)$. The generators of the fundamental group of the ends ϵ_3, ${\epsilon'}_3$ spinning asymptotically along to each Reeb component are also homotopic to β, respectively β', so they are trivial as well. It is easy to see that the loops C_{1n} and C_{2n} are freely homotopic to θ for all $n \in \mathbb{N}$.

It remains to see that the loops indicated in Figure 8 below are also trivial. One can see in fact that the loop which passes from a "tube" C_{1i} to the "tube" $C_{1(i+k)}$ is freely homotopic to $\beta^k.\beta'^k$, while that which passes from a "tube" C_{2i} to the "tube" $C_{2(i+k)}$ is freely homotopic to $\alpha^k.\alpha'^k$, hence trivial in M.

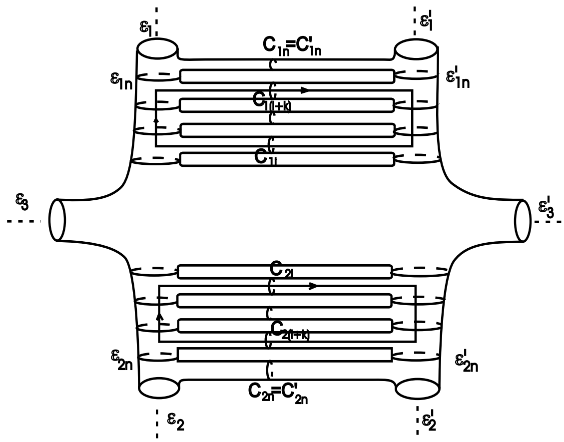


Figure 8:

We note that if we make this surgery along g closed transversal curves, homotopic to the factor S^1 of $S^1 \times S^2$, we obtain a foliation of $S^1 \times S^2$ which satisfies P_2 and has as leaf a compact surface Σ_g of genus g, and g Reeb components. Denote this foliation by $\mathcal{F}$, and for an integer $n \geq 1$, let us take a closed simple transversal curve which makes n turns along the factor S^1. The associated finite cyclic covering gives us a foliation of $S^1 \times S^2$, which still satisfies P_2 and contains n compact surfaces Σ_g as leaves. If one would have taken dense foliations by planes of $T^2 \times [0, 1]$ at the beginning, this construction would have given a foliation of $S^1 \times S^2$ with locally dense leaves of infinite genus.

To close this section, we note that, in all examples above, the fundamental group of the graph Y is injectively mapped into the fundamental group of the manifold.

4 Property P_1 and foliations with nonexponential growth

In this Section, M^3 is a closed orientable manifold and $\mathcal{F}$ is a transversely orientable codimension one foliation of class at least C^2, with nonexponential growth. The goal here is to prove Theorem 1. We recall some definitions and results about foliations with nonexponential growth ([C.C3]).

Any compact leaf L of $\mathcal{F}$ will be called of *level zero* ; a leaf L of $\mathcal{F}$ is of *level* $k > 0$ if $\overline{L} - L$ contains only leaves of level at most $k - 1$ and at least one leaf of level $k - 1$. The *substructure* $S(L)$ of L is the union of all leaves of $\overline{L}$ of level strictly lower than the level of L. If each leaf of $S(L)$ is proper, we say that L has a *totally proper substructure* . If, in addition, L is proper, we say that L is a *totally proper leaf* . If L has nonexponential growth then the substructure $S(L)$ of L is totally proper.

Proof of the Theorem 1. First of all we shall prove the assertion about the depth of $\mathcal{F}$. If $\mathcal{F}$ does not have a compact leaf, then it follows by Plante's result that $\mathcal{F}$ is without holonomy [P]. Therefore, by property P_1, the leaves are planes and $M = T^3$ [R.R]. Suppose that the set C of compact leaves is not empty and let L' be a non compact leaf of $\mathcal{F}$ and $L \subset S(L')$. We claim that L is compact. Indeed, let us suppose that this is not the case, and let T be a compact leaf in $\overline{L}$. Fix x_0 in T and let $\gamma : (-\epsilon, \epsilon) \to M$ with $\gamma(0) = x_0$ be an arc transverse to the foliation. Property P_1 implies that the holonomy representation $\phi : \pi_1(T) \to \mathrm{Diff}((-\epsilon, \epsilon), 0)$ of T is injective. By Reeb's Theorem, its genus $g(T) \geq 1$. If $g(T) > 1$ then the holonomy group of T has exponential growth. Thus we have $g(T) = 1$. Since L is totally proper, the lateral holonomy of T in the side which L approaches is cyclic. Then there exists $\psi : T^2 \times [0, 1] \to M$ such that $\psi(T^2 \times \{0\}) = T$, $\psi(T^2 \times \{1\})$ is transverse to $\mathcal{F}$ and the leaves of $\mathcal{F}|_{\psi(T^2 \times (0,1))}$ are half-cylinders which spiral around T. These half-cylinders are ends of open leaves of $\mathcal{F}$. Since for each one of these ends the generator of the fundamental group of the end has trivial holonomy, the property P_1 implies that they are trivial in the fundamental group of the leaf. Therefore these leaves, in particular L and L', are proper planes that accumalate in T, which is a contradiction with the fact that $L \subset S(L')$. So, our claim is proved.

Since the substructure $S(L)$ for any leaf is a union of compact leaves, every noncompact leaf has trivial holonomy; by property P_1, they are planes and the compact leaves are torus, each one with holonomy of rank two. However, each model is a manifold with boundary, foliated by planes and tori tangent to the boundary. According to the classification in [R.R.], each model is homeomorphic either to $T^2 \times [0, 1]$ with a transverse foliation to the factor $[0, 1]$, as in Example 1, or to $D^2 \times S^1$ with a Reeb foliation. Thus, if $\mathcal{F}$ does not have a Reeb component, all models are of type 1 and, since M is closed, it is a fibration over S^1 with fiber T^2 foliated as in Example 1.

Let us note that, in the other case, the single models of type 2 are the Reeb components; thus each vertex of the associated graph Y is attached to a single edge; as a consequence Y consists of two vertices and one edge. This edge corresponds either to a common boundary of two models of type 2, or a submanifold of M homeomorphic to $T^2 \times [0, 1]$ with a transverse foliation to the factor [0, 1], as in Example 1. In the first case, since the holonomy of the compact leaf is abelian of rank two, $(M, \mathcal{F})$ is obtained by gluing two copies of $D^2 \times S^1$ with Reeb foliations, by a diffeomorphism of T^2 which keeps this property. In the second case, since each boundary component of $T^2 \times [0, 1]$ has abelian lateral holonomy of rank two, the two Reeb components can be glued to each one of these components by any diffeomorphism of T^2. □

We note that if $\mathcal{F}$ has exponential growth and satisfies property P_1, then the problem becomes nontrivial. We do not even know if the surfaces Σ_2 can be leaves of such foliations. The following question has been posed by Etienne Ghys and Takashi Tsuboi:

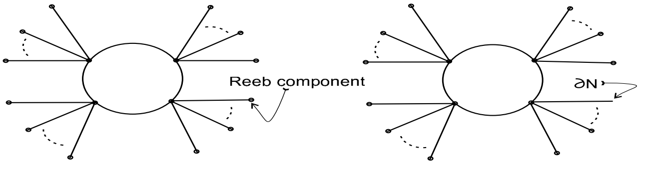
Question 1. *Does exist an injective representation of* $\pi_1(\Sigma_2)$ *in the group of diffeomorphisms* $\mathrm{Diff}^r(\mathbb{R}, 0)$ *(*$r \geq 1$*) of* $\mathbb{R}$ *which fix zero?*

5 Foliations satisfying $\pi_1(B\Gamma) = \pi_1(M)$

We will show Theorem 2 which indicates that is not probable that exists a classification of the foliated manifolds satisfying P_2.

Proof of Theorem 2. Let $(V, \mathcal{F}')$ be the foliated manifold of Example 4, such that the associated graph is as in figure 6a. Let us consider $S^1 \times S^2$ with the foliation with k surfaces Σ_g as leaves, as in the Example 5.

We see that the associated graph is as in figure 9a below. Let us remove the interior of one of the Reeb components. The manifold N obtained has a toric boundary component and the graph is as in figure 9b.

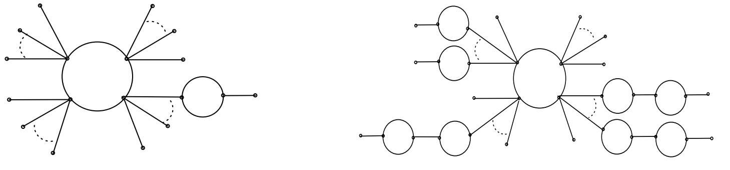


(a) (b)

Figure 9:

Now we glue V and N by their boundaries changing meridian by parallels where the meridian of ∂N is the nul homotopic curve in the Reeb component removed. To see that the obtained foliated manifold W satisfies P_2 it is enough to show that any loop in N which is trivial in the Reeb component removed is also trivial in W. These loops are freely homotopic to the meridian (or powers of the meridian) of ∂N which, after gluing, corresponds to the parallel of ∂V, and hence it is trivial in W.

The graph Y associated to W is as in figure 10a, and we can repeat the surgery along each Reeb component to obtain the foliated manifold W' which satisfies P_2, and whose graph is as in figure 10b.



(a) (b)

Figure 10:

Now, for each new Reeb component, we can repeat the operation until we obtain the desired manifold M, whose graph Y has n cycles. It is clear, by construction, that $\pi_1(Y) = \mathbb{F}_n$ is injected into $\pi_1(M)$. By construction, this foliation is almost without holonomy, and thus with polynomial growth [Hec]. In [C.C4] the authors build on $\Sigma_2 \times [0, 1]$ foliations with each types of growth a), b), c) or d). These constructions are valid for $\Sigma_g \times [0, 1]$ and thus we can replace in M the compact leaves Σ_g by a product $\Sigma_g \times [0, 1]$ provided with foliations with each one of types of growth a), b), c), or d). Since the leaves Σ_g are totally compressible in M, it is clear that the obtained foliated manifold satisfies property P_2. □

Corollary 1. *Let $(M^3, \mathcal{F})$ be a foliated manifold which satisfies property P_2 and let Y be the associated graph, with $\pi_1(Y) = \mathbb{F}_n$ (where for n=0, Y is a tree and for n=1, $\mathbb{F}_1 = \mathbb{Z}$). Then, for all $k \in \mathbb{N}$, one can modify $(M^3, \mathcal{F})$ by a surgery in order to obtain a manifold $(M', \mathcal{F}')$ which still satisfies P_2, and such that the associated graph Y' has as its fundamental group the free group $\mathbb{F}_{n+k}$.*

Proof. If $\mathcal{F}$ has no Reeb component, we saw that M is a fibration over S^1 with fiber T^2 foliated as in Example 1. Let γ be a transversal loop in the interior of one of the connected components of the complement of the compact leaves, and N the manifold with boundary obtained by Reeb surgery along γ. Now glue N with the manifold V of Example 4 by their boundaries, changing meridians by parallels. In the same way as in the proof of Theorem 2, the foliated manifold $(M_1, \mathcal{F}_1)$ obtained satisfies property P_2, and the associated graph has as fundamental group the free group with two generators $\mathbb{F}_2$. Now let us remove the interior of the Reeb component of $\mathcal{F}_1$ and by the same way let us glue its border with ∂V. We thus obtain a manifold $(M_2, \mathcal{F}_2)$ which satisfies P_2, and its associated graph has as fundamental group the free group with three generators $\mathbb{F}_3$. By repeating this operation $k-2$ times, we obtain $(M', \mathcal{F}')$, which satisfies P_2 and the associated graph has as fundamental group the free group $\mathbb{F}_{k+1}$.

Now, if $\mathcal{F}$ has Reeb components, we makes this operation k times, on the basis of one of the Reeb components to obtain the desired manifold $(M', \mathcal{F}')$. □

If $\mathcal{F}$ has nonexponential growth we ask the following questions:

Question 3. *What is the fundamental group of the foliated 3-manifolds satisfying property P_2 ?*

Question 4. *If $\mathcal{F}$ is a C^2 transversely orientable codimension one foliation of a closed manifold of dimension n, what is the fundamental group of $B\Gamma$?*

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